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Bethe ansatz equations for open spin chains from giant gravitons

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Abstract

We investigate the open spin chain describing the scalar sector of the $Y = 0$ giant graviton brane at weak coupling. We provide a direct proof of integrability in the $SU(2)$ and $SU(3)$ sectors by constructing the transfer matrices. We determine the eigenvalues of these transfer matrices in terms of roots of the corresponding Bethe ansatz equations (BAEs). Based on these results, we propose BAEs for the full $SO(6)$ sector. We find that, in the weak-coupling limit, the recently-proposed all-loop BAEs essentially agree with those proposed in the present work.

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1 Introduction

Integrability of planar $\mathcal{N} = 4$ Yang-Mills has been investigated primarily for closed spin chains associated with long single-trace operators. Indeed, the anomalous dimensions of these operators are given by a set of closed-chain Bethe ansatz equations (BAEs) [1]-[3], which can be derived from the all-loop bulk S-matrix [4]-[8]. (These results are valid only asymptotically. For operators of finite length, there are finite-size corrections. See e.g. [9] and references therein.)

Nevertheless, progress is also being made in understanding open spin chains associated with certain determinant-like operators [10] corresponding to open strings attached to maximal giant gravitons [11]. This problem was first considered by Berenstein and Vázquez [12], who determined the open-chain Hamiltonian describing the mixing of such operators at one loop. They also computed the one-loop boundary S-matrix, and argued that the Hamiltonian is integrable. This work was extended to two loops by Agarwal [13] (see also [14]) and by Hofman and Maldacena [15]. The latter also proposed all-loop boundary S-matrices, up to scalar factors which were subsequently found in [16, 17]. These boundary S-matrices were further investigated in [18]-[20]. [1]

Although Berenstein and Vázquez [12] found an integrable Hamiltonian, they did not give the corresponding open-chain BAEs. Remarkably, this problem (which is the open-chain version of the problem solved in the pioneering work of Minahan and Zarembo [1]) has remained unsolved for more than four years. One of the aims of this paper is to determine these BAEs, whose solutions give the anomalous dimensions of the determinant-like operators. (As in the periodic case, these results are expected to be valid only asymptotically.) Another aim of this work is to construct and diagonalize the corresponding commuting transfer matrix (i.e., the generating functional for the Hamiltonian and higher local conserved charges), which would provide a direct proof of the model's integrability. Moreover, we would like to compare the one-loop BAEs with the interesting all-loop BAEs which have recently been derived by Galleas [26] (see also [27]-[29]) from the all-loop bulk and boundary S-matrices.

Berenstein and Vázquez restricted their attention to the scalar sector of $\mathcal{N} = 4$ Yang-Mills, which has $SO(6)$ symmetry. In terms of the complex scalar fields W, Z, Y, Hofman and Maldacena [15] considered operators with a large number of Z's, and distinguished two cases of interest :

[1] For further applications of integrable open spin chains in gauge theory and string theory, see e.g. [21]-[25] and references therein.

$Y = 0$ **brane:** The vacuum corresponds to the operator

$$\epsilon_{i_1 \cdots i_N}^{j_1 \cdots j_N} Y_{j_1}^{i_1} \cdots Y_{j_{N-1}}^{i_{N-1}} (Z \cdots Z)_{j_N}^{i_N} ; \tag{1.1}$$

and excitations correspond to replacing some of the Z's by impurities, e.g.,

$$\epsilon_{i_1 \cdots i_N}^{j_1 \cdots j_N} Y_{j_1}^{i_1} \cdots Y_{j_{N-1}}^{i_{N-1}} (Z \cdots Z \chi Z \cdots Z)_{j_N}^{i_N} . \tag{1.2}$$

The sets of fields inside $(\cdots)_{j_N}^{i_N}$ constitute the states of the open spin chain. The fields at the boundaries of this chain cannot be Y's, since the operator would then factorize into a determinant and a single trace, and therefore would not describe an open string.

$Z = 0$ **brane:** The vacuum corresponds to the operator

$$\epsilon_{i_1 \cdots i_N}^{j_1 \cdots j_N} Z_{j_1}^{i_1} \cdots Z_{j_{N-1}}^{i_{N-1}} (\chi Z \cdots Z \chi')_{j_N}^{i_N} , \tag{1.3}$$

where χ and χ' are boundary degrees of freedom; and excitations correspond to replacing some of the Z's by impurities, e.g.,

$$\epsilon_{i_1 \cdots i_N}^{j_1 \cdots j_N} Z_{j_1}^{i_1} \cdots Z_{j_{N-1}}^{i_{N-1}} (\chi Z \cdots Z \chi'' Z \cdots Z \chi')_{j_N}^{i_N} . \tag{1.4}$$

For simplicity, we consider in this paper only the former case of the $Y = 0$ giant graviton brane. Owing to the difficulty of treating directly the full $SO(6)$ scalar sector, we instead proceed by first examining simpler subsectors, namely, $SU(2)$ and $SU(3)$. We then use the results for these subsectors to conjecture BAEs for the full $SO(6)$ sector. Finally, we compare these one-loop BAEs with the weak-coupling limit of the all-loop BAEs [26]. We find that these two sets of results essentially agree – the only mismatch is in the exponent of the term corresponding to the "massive" node, which also occurs in the periodic case. [2]

The outline of this paper is as follows. In Sec. 2 we consider the $SU(3)$ sector. We construct the transfer matrix and determine its eigenvalues in terms of roots of corresponding BAEs. Based on these results, and on the results for the $SU(2)$ sector which we briefly discuss in Appendix A, we propose in Sec. 3 BAEs for the full $SO(6)$ scalar sector. In Sec. 4 we perform the comparison with the all-loop BAEs. In Sec. 5 we briefly discuss these results, and list some interesting related problems which remain unsolved. In Appendix B we present some numerical results which demonstrate the completeness of our Bethe ansatz solution in the $SU(3)$ sector for spin chains of short length.

2 The $SU(3)$ sector

We consider in this section, as in Section 4.3.1 of [15], the subsector $SU(3) \subset SO(6)$.

[2] In earlier versions of our paper, we pointed out some errors in the first version of [26], which were subsequently corrected in the second (published) version.

2.1 Hamiltonian

In the $SU(3)$ sector, the Hilbert space is [3]

$$\overset{0}{\underset{\downarrow}{}}\!\!\!C^2 \otimes \overset{1}{\underset{\downarrow}{}}\!\!\!C^3 \otimes \cdots \overset{L}{\underset{\downarrow}{}}\!\!\!C^3 \otimes \overset{L+1}{\underset{\downarrow}{}}\!\!\!C^2 \ . \tag{2.1}$$

The one-loop mixing matrix for operators of the type (1.2) is the open-chain Hamiltonian given by (cf. Eq. (2.15) in [12])

$$H = 2g^2 \left(Q_0^Y h_{0,1} Q_0^Y + \sum_{l=1}^{L-1} h_{l,l+1} + Q_{L+1}^Y h_{L,L+1} Q_{L+1}^Y \right) , \tag{2.2}$$

where

$$g^2 = \frac{\lambda}{16\pi^2} , \tag{2.3}$$

and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. The two-site Hamiltonian $h_{l,l+1}$ is given by

$$h_{l,l+1} = \mathbb{I}_{l,l+1} - \mathcal{P}_{l,l+1} , \tag{2.4}$$

where $\mathbb{I}$ and $\mathcal{P}$ are the identity and permutation matrices, respectively. The latter can be expressed as

$$\mathcal{P} = \sum_{a,b=1}^{n} e_{ab} \otimes e_{ba} , \tag{2.5}$$

where e_{ab} is the usual elementary $n \times n$ matrix whose (a, b) matrix element is 1, and all others are zero; and here $n = 3$. Moreover, we take Q^Y to be the projector [4]

$$Q^Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{2.6}$$

which is used to implement the restriction (noted below (1.2)) that Y cannot appear at sites 0 and $L + 1$. We drop the null rows and columns of the left and right boundary terms in the Hamiltonian,

$$H_{bt}^L = Q_0^Y h_{0,1} Q_0^Y , \qquad H_{bt}^R = Q_{L+1}^Y h_{L,L+1} Q_{L+1}^Y . \tag{2.7}$$

[3] Following [15], we define the origin of the spin chain at site 0 instead of site 1.

[4] We choose the basis

$$|W\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad |Z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad |Y\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

Hence, these boundary terms should be understood as 6×6 matrices acting on $C^2\otimes C^3$ and $C^3\otimes C^2$, respectively. We observe that the boundary terms (2.7) have the symmetry

$$\left[H^L_{bt}\,,\mathfrak{h}_0\,\mathfrak{g}_1\right]=0\,,\qquad \left[H^R_{bt}\,,\mathfrak{g}_L\,\mathfrak{h}_{L+1}\right]=0\,, \tag{2.8}$$

where

$$\mathfrak{g}=\left(\begin{array}{cc}\mathfrak{h} & 0\\ 0 & e^{i\theta}\end{array}\right)\,,\qquad \mathfrak{h}\in SU(2)\,. \tag{2.9}$$

2.2 Transfer matrix

We now proceed to construct the commuting open-chain transfer matrix which contains the Hamiltonian (2.2). Following the general recipe of Sklyanin [30], there are two main ingredients: R-matrices and K-matrices. The R-matrix is a solution $R(u)$ of the Yang-Baxter equation (YBE)

$$R_{12}(u_1-u_2)\,R_{13}(u_1)\,R_{23}(u_2)=R_{23}(u_2)\,R_{13}(u_1)\,R_{12}(u_1-u_2)\,. \tag{2.10}$$

For the case at hand with $SU(3)$ symmetry, the R-matrix is well known to be the 9×9 matrix acting on $C^3\otimes C^3$ given by

$$R(u)=u\mathbb{I}+i\mathcal{P}\,. \tag{2.11}$$

Here the right K-matrix $K^R(u)$ is a 6×6 matrix acting on $C^3\otimes C^2$ which is a solution of the right boundary Yang-Baxter equation (BYBE) [31, 32]

$$\begin{aligned}&R_{12}(u_1-u_2)\,K^R_{13}(u_1)\,R_{12}(u_1+u_2)\,K^R_{23}(u_2)\\ &\quad=K^R_{23}(u_2)\,R_{12}(u_1+u_2)\,K^R_{13}(u_1)\,R_{12}(u_1-u_2)\,.\end{aligned} \tag{2.12}$$

Assuming that $K^R(u)$ has the same symmetry as the right boundary term H^R_{bt} (2.8), i.e.,

$$\left[K^R(u)\,,\mathfrak{g}\otimes\mathfrak{h}\right]=0\,, \tag{2.13}$$

leads to the ansatz

$$K^R(u)=\left(\begin{array}{cccccc}a_1(u)+a_2(u) &&&&&\\ & a_1(u) & a_2(u) &&&\\ & a_2(u) & a_1(u) &&&\\ &&& a_1(u)+a_2(u) &&\\ &&&& a_3(u) &\\ &&&&& a_3(u)\end{array}\right)\,, \tag{2.14}$$

where matrix elements which are zero are left empty. The boundary Yang-Baxter equation (2.12) together with the regularity condition $K^R(0) = \mathbb{I}$ imply [5]

$$a_1(u) = 1 - u^2\,, \qquad a_2(u) = -2iu\,, \qquad a_3(u) = 1 + u^2\,. \tag{2.15}$$

This K-matrix has the feature that its first derivative evaluated at $u = 0$ is proportional to the right boundary term,

$$\frac{d}{du} K^R(u)\Big|_{u=0} = 2i\left(H^R_{bt} - \mathbb{I}\right)\,, \tag{2.16}$$

up to an additive term proportional to the identity.

The left K-matrix $K^L(u)$ is a 6×6 matrix acting on $C^2 \otimes C^3$ which is a solution of the left BYBE

$$\begin{aligned} &R_{12}(-u_1 + u_2)\, K^L_{31}(u_1)^{t_1}\, R_{12}(-u_1 - u_2 - \eta)\, K^L_{32}(u_2)^{t_2} \\ &\quad = K^L_{32}(u_2)^{t_2}\, R_{12}(-u_1 - u_2 - \eta)\, K^L_{31}(u_1)^{t_1}\, R_{12}(-u_1 + u_2)\,, \end{aligned} \tag{2.17}$$

where t_i denotes transposition in the i^{th} space, and $\eta = 3i$ appears in the crossing-unitarity relation

$$R_{12}(u)^{t_1}\, R_{12}(-u - \eta)^{t_1} \propto \mathbb{I}\,, \tag{2.18}$$

where the proportionality factor is some scalar function of u. Assuming that $K^L(u)$ has the same symmetry as the left boundary term H^L_{bt} (2.8), i.e.,

$$\left[K^L(u)\,, \mathfrak{h} \otimes \mathfrak{g}\right] = 0\,, \tag{2.19}$$

leads to the ansatz

$$K^L(u) = \begin{pmatrix} b_1(u) + b_2(u) & & & & & \\ & b_1(u) & & b_2(u) & & \\ & & b_3(u) & & & \\ & b_2(u) & & b_1(u) & & \\ & & & & b_1(u) + b_2(u) & \\ & & & & & b_3(u) \end{pmatrix}. \tag{2.20}$$

We find the following solution of the left BYBE (2.17)

$$b_1(u) = iu + u^2\,, \qquad b_2(u) = -3 + 2iu\,, \qquad b_3(u) = -2iu - u^2\,. \tag{2.21}$$

[5] There is in fact a one-parameter family of such solutions. We fix the parameter so as to match with the boundary term (2.16).

This K-matrix has the feature that its value at $u = 0$ is proportional to the left boundary term,

$$K^L(0) = 3\left(H^L_{bt} - \mathbb{I}\right)\,, \tag{2.22}$$

up to an additive term proportional to the identity.

The transfer matrix $t(u)$ is given by

$$t(u) = \mathrm{tr}_a\, K^L_{0a}(u)\, T_{a1\cdots L}(u)\, K^R_{aL+1}(u)\, \hat{T}_{a1\cdots L}(u)\,, \tag{2.23}$$

where the trace (tr) is over a 3-dimensional auxiliary space denoted by a. The argument of the trace acts on

$$\overset{\overset{0}{\downarrow}}{C^2} \otimes \overset{\overset{a}{\downarrow}}{C^3} \otimes \overset{\overset{1}{\downarrow}}{C^3} \otimes \cdots \overset{\overset{L}{\downarrow}}{C^3} \otimes \overset{\overset{L+1}{\downarrow}}{C^2}\,, \tag{2.24}$$

and therefore $t(u)$ acts on (2.1), as does the Hamiltonian. The monodromy matrices T and $\hat{T}$ are given by

$$T_{a1\cdots L}(u) = R_{a1}(u)\cdots R_{aL}(u)\,, \qquad \hat{T}_{a1\cdots L}(u) = R_{aL}(u)\cdots R_{a1}(u)\,. \tag{2.25}$$

Indeed, it can be shown along the lines [30] that the transfer matrix (2.23) obeys the fundamental commutativity property

$$[t(u)\,, t(v)] = 0\,, \tag{2.26}$$

by virtue of the fact that the R and K matrices obey their respective YBEs (2.10), (2.12), (2.17). In fact, the latter equation for $K^L(u)$ was engineered to ensure this commutativity. It can also be shown that this transfer matrix contains the Hamiltonian (2.2),

$$H = c_1 \frac{d}{du} t(u)\Big|_{u=0} + c_2 \mathbb{I}\,, \tag{2.27}$$

where

$$c_1 = 2g^2\left(\frac{i}{6}(-1)^L\right)\,, \qquad c_2 = 2g^2\left(L + \frac{4}{3}\right)\,. \tag{2.28}$$

The relations (2.26) - (2.28) provide a direct proof of the integrability of the Hamiltonian.

We observe that the transfer matrix has the $SU(2)\times U(1)$ symmetry

$$\left[t(u)\,, \mathfrak{h}\otimes\mathfrak{g}^{\otimes L}\otimes\mathfrak{h}\right] = 0\,, \tag{2.29}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are defined in (2.9). The eigenstates of the transfer matrix therefore form representations of $SU(2)$.

2.3 Bethe ansatz

The commutativity property (2.26) implies that it is possible to find eigenstates $|\Lambda\rangle$ of the transfer matrix $t(u)$ which are independent of u,

$$t(u)\,|\Lambda\rangle = \Lambda(u)\,|\Lambda\rangle\,. \tag{2.30}$$

We now proceed to determine the eigenvalues $\Lambda(u)$ by the analytical Bethe ansatz [33]-[36].

Acting with the transfer matrix on the vacuum state (1.1), i.e.,

$$|Z\cdots Z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \cdots \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.31}$$

we find that the vacuum eigenvalue is given by

$$\Lambda_0(u) = -\frac{(u+i)}{(2u+i)}\left[(2u+3i)(u+i)^{2L+3} + 4(u+i)u^{2L+3}\right]\,. \tag{2.32}$$

Known results for the closed and open $SU(3)$ chains (see, e.g., [37, 38]) suggest that a general eigenvalue should have the "dressed" form

$$\begin{aligned} \Lambda(u) &= -\frac{(u+i)}{(2u+i)}\Bigg\{(2u+3i)(u+i)^{2L+3}\frac{Q_1(u-i/2)}{Q_1(u+i/2)} \\ &+ u^{2L+3}\left[B_1(u)\frac{Q_1(u+3i/2)}{Q_1(u+i/2)}\frac{Q_2(u)}{Q_2(u+i)} + B_2(u)\frac{Q_2(u+2i)}{Q_2(u+i)}\right]\Bigg\}, \end{aligned} \tag{2.33}$$

where

$$Q_j(u) = \prod_{k=1}^{m_j}(u-u_{j,k})(u+u_{j,k})\,, \qquad j=1\,,2\,, \tag{2.34}$$

and

$$B_1(u) + B_2(u) = 4(u+i)\,. \tag{2.35}$$

By considering the $L=1$ case, we readily determine

$$B_1(u) = 2u+3i\,, \qquad B_2(u) = 2u+i\,. \tag{2.36}$$

The BAEs for the zeros $u_{j,k}$ of the functions $Q_j(u)$ (2.34) follow the fact that $\Lambda(u)$ in (2.33) is analytic at $u = u_{1,k} - i/2$ and at $u = u_{2,k} - i$. In terms of the standard notation

$$e_n(u) = \frac{u + in/2}{u - in/2}\,, \tag{2.37}$$

the BAEs take the form

$$
\begin{aligned}
e_1(u_{1,k})^{2L+2} &= \prod_{\substack{j=1\\ j\neq k}}^{m_1} e_2(u_{1,k}-u_{1,j})\, e_2(u_{1,k}+u_{1,j}) \\
&\quad \times \prod_{j=1}^{m_2} e_{-1}(u_{1,k}-u_{2,j})\, e_{-1}(u_{1,k}+u_{2,j})\,, \quad k=1\,,\ldots\,,m_1\,, \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{m_2} e_2(u_{2,k}-u_{2,j})\, e_2(u_{2,k}+u_{2,j}) \\
&\quad \times \prod_{j=1}^{m_1} e_{-1}(u_{2,k}-u_{1,j})\, e_{-1}(u_{2,k}+u_{1,j})\,, \quad k=1\,,\ldots\,,m_2\,.
\end{aligned}
\tag{2.38}
$$

As a check, we observe that reducing to the $SU(2)$ subsector by removing the Bethe roots $\{u_{2,k}\}$ yields exactly the same result derived in Appendix A, namely, Eq. (A.11).

In view of the relation (2.27) between the transfer matrix and the Hamiltonian, we find that the eigenvalues of the Hamiltonian (2.2) are given by

$$
E = c_1 \frac{d}{du}\Lambda(u)\Big|_{u=0} + c_2 = 2g^2 \sum_{k=1}^{m_1} \frac{1}{u_{1,k}^2+1/4}\,. \tag{2.39}
$$

The problem of determining the one-loop anomalous dimensions of operators of the type (1.2) in the $SU(3)$ sector is in principle solved by (2.38), (2.39).

We have verified the completeness of this solution for $L=1,2,3$, as discussed in Appendix B. We observe that for $L>1$, the numbers of Bethe roots for a given eigenvalue satisfy

$$
0 \le m_1 \le L\,, \qquad 0 \le m_2 \le m_1\,. \tag{2.40}
$$

Degenerate states $|\Lambda\rangle$ with the same eigenvalue $\Lambda(u)$ (characterized by given sets of Bethe roots $\{u_{1,k}\}\,,\{u_{2,k}\}$) do not necessarily form irreducible representations of $SU(2)$; they are characterized by one or more values of the $SU(2)$ spin s. The following inequality appears to hold

$$
s \le \frac{1}{2}\left(L+2-m_1-m_2\right)\,. \tag{2.41}
$$

3 The $SO(6)$ sector

We have not formulated the transfer matrix for the full $SO(6)$ scalar sector, for which the Hamiltonian is given by Eq. (2.15) in [12]. However, based on our results for the $SU(3)$

and $SU(2)$ sectors presented in Sec. 2 and Appendix A, respectively, it is not difficult to conjecture the result for the BAEs. Indeed, since $SO(6) \approx SU(4)$ has rank three, we expect that the (one-loop) BAEs are given by

$$\begin{aligned} e_1(u_{1,k})^{2L+2} &= \prod_{\substack{j=1\\j\neq k}}^{m_1} e_2(u_{1,k}-u_{1,j})\, e_2(u_{1,k}+u_{1,j}) \\ &\times \prod_{j=1}^{m_2} e_{-1}(u_{1,k}-u_{2,j})\, e_{-1}(u_{1,k}+u_{2,j}) \\ &\times \prod_{j=1}^{m_3} e_{-1}(u_{1,k}-u_{3,j})\, e_{-1}(u_{1,k}+u_{3,j})\,, \quad k=1\,,\ldots\,,m_1\,, \end{aligned}$$

$$\begin{aligned} 1 &= \prod_{\substack{j=1\\j\neq k}}^{m_2} e_2(u_{2,k}-u_{2,j})\, e_2(u_{2,k}+u_{2,j}) \\ &\times \prod_{j=1}^{m_1} e_{-1}(u_{2,k}-u_{1,j})\, e_{-1}(u_{2,k}+u_{1,j})\,, \quad k=1\,,\ldots\,,m_2\,, \end{aligned}$$

$$\begin{aligned} 1 &= \prod_{\substack{j=1\\j\neq k}}^{m_3} e_2(u_{3,k}-u_{3,j})\, e_2(u_{3,k}+u_{3,j}) \\ &\times \prod_{j=1}^{m_1} e_{-1}(u_{3,k}-u_{1,j})\, e_{-1}(u_{3,k}+u_{1,j})\,, \quad k=1\,,\ldots\,,m_3\,. \end{aligned} \tag{3.1}$$

Reducing to the $SU(3)$ subsector by removing the Bethe roots $\{u_{3,k}\}$ yields (2.38), and reducing further to the $SU(2)$ subsector by also removing the roots $\{u_{2,k}\}$ yields (A.11). The open-chain BAEs (3.1) are essentially "doubled" with respect to the corresponding closed-chain results of Minahan and Zarembo [1], except for the exponent on the LHS of the first equation, which is $2L+2$ for the open chain with $L+2$ sites, and is L for the closed chain with L sites.

4 Comparison with all-loop BAEs

All-loop BAEs for the $Y=0$ brane have recently been proposed in an interesting recent paper by Galleas [26]. We now wish to compare those equations with the one-loop BAEs which we have proposed for the scalar sector. This will require performing the weak-coupling limit of the former BAEs, and then reducing to the scalar sector.

4.1 All-loop BAEs

We begin by recalling the all-loop BAEs from [26]: [6]

$$\left[\frac{x_k^+}{x_k^-}\right]^{(-2L-2N+m_1+m_2)}\Phi(\lambda_k) = \prod_{\substack{j=1\\j\neq k}}^{N}\left[S_0(\lambda_k,\lambda_j)S_0(\lambda_j,-\lambda_k)\frac{(x_k^-+x_j^-)(x_k^--x_j^+)}{(x_k^+-x_j^-)(x_k^++x_j^+)}\right]^2 \times \prod_{\alpha=1}^{2}\prod_{l=1}^{m_\alpha}\frac{(x_k^+-z_{\alpha,l}^-)(x_k^++z_{\alpha,l}^-)}{(x_k^--z_{\alpha,l}^-)(x_k^-+z_{\alpha,l}^-)} \tag{4.1}$$

$$\prod_{j=1}^{N}\frac{(z_{\alpha,k}^-+x_j^-)}{(z_{\alpha,k}^--x_j^-)}\frac{(z_{\alpha,k}^--x_j^+)}{(z_{\alpha,k}^-+x_j^+)}\Theta(z_{\alpha,k}^\pm) = \prod_{j=1}^{n_\alpha}\frac{(z_{\alpha,k}^-+\frac{1}{z_{\alpha,k}^-}-\tilde\lambda_{\alpha,j}-\frac{i}{2g})}{(z_{\alpha,k}^-+\frac{1}{z_{\alpha,k}^-}-\tilde\lambda_{\alpha,j}+\frac{i}{2g})}\frac{(z_{\alpha,k}^-+\frac{1}{z_{\alpha,k}^-}+\tilde\lambda_{\alpha,j}-\frac{i}{2g})}{(z_{\alpha,k}^-+\frac{1}{z_{\alpha,k}^-}+\tilde\lambda_{\alpha,j}+\frac{i}{2g})} \qquad \alpha=1,2 \qquad k=1,\ldots,m_\alpha \tag{4.2}$$

$$\prod_{j=1}^{m_\alpha}\frac{(\tilde\lambda_{\alpha,k}-z_{\alpha,j}^--\frac{1}{z_{\alpha,j}^-}+\frac{i}{2g})}{(\tilde\lambda_{\alpha,k}-z_{\alpha,j}^--\frac{1}{z_{\alpha,j}^-}-\frac{i}{2g})}\frac{(\tilde\lambda_{\alpha,k}+z_{\alpha,j}^-+\frac{1}{z_{\alpha,j}^-}+\frac{i}{2g})}{(\tilde\lambda_{\alpha,k}+z_{\alpha,j}^-+\frac{1}{z_{\alpha,j}^-}-\frac{i}{2g})} = \prod_{\substack{j=1\\j\neq k}}^{n_\alpha}\frac{(\tilde\lambda_{\alpha,k}-\tilde\lambda_{\alpha,j}+\frac{i}{g})}{(\tilde\lambda_{\alpha,k}-\tilde\lambda_{\alpha,j}-\frac{i}{g})}\frac{(\tilde\lambda_{\alpha,k}+\tilde\lambda_{\alpha,j}+\frac{i}{g})}{(\tilde\lambda_{\alpha,k}+\tilde\lambda_{\alpha,j}-\frac{i}{g})} \qquad \alpha=1,2 \qquad k=1,\ldots,n_\alpha\,, \tag{4.3}$$

where $\Theta(z^\pm)$ is given by

$$\Theta(z^\pm) = \frac{2z^+z^-(z^++\frac{1}{z^+}-\frac{i}{2g})}{(z^++z^-)(z^+z^-+1)}\,. \tag{4.4}$$

Moreover, $\Phi(\lambda)$ is given by [26]

$$\Phi(\lambda) = \left[\left(\frac{x^+}{x^-}\right)^2\frac{1}{k_0^+(-\lambda)k_0^-(\lambda)}\right]^2, \tag{4.5}$$

where $S_0(\lambda,\lambda')$ and $k_0^\pm(\lambda)$ are the scalar factors of the bulk and boundary S-matrices, respectively.

Assuming that $z^\pm$ satisfy the usual constraint

$$z^++\frac{1}{z^+}-z^--\frac{1}{z^-} = \frac{i}{g}\,, \tag{4.6}$$

[6] In the first version of [26], the third set of BAEs (here, (4.3)) contained some sign errors, and the expression for $\Phi(\lambda)$ differed from (4.5).

we find that the quantity (4.4) simplifies to unity,

$$\Theta(z^{\pm}) = 1\,. \tag{4.7}$$

Hence, although the all-loop BAEs seem to depend on both z^+ and z^-, they can in fact be expressed in terms of z^- alone.

In order to bring these equations to a more familiar form, we perform the following identifications [7] [7]

$$\begin{aligned}
x_j^{\pm} &= \frac{x_{4,j}^{\pm}}{g}\,, \qquad j = 1,\ldots,K_4 \equiv N\,,\\
z_{1,j}^{-} &= \frac{g}{x_{1,j}}\,, \qquad j = 1,\ldots,K_1\,,\\
z_{1,K_1+j}^{-} &= \frac{x_{3,j}}{g}\,, \qquad j = 1,\ldots,K_3\,, \qquad m_1 \equiv K_1 + K_3\,,\\
z_{2,j}^{-} &= \frac{x_{5,j}}{g}\,, \qquad j = 1,\ldots,K_5\,,\\
z_{2,K_5+j}^{-} &= \frac{g}{x_{7,j}}\,, \qquad j = 1,\ldots,K_7\,, \qquad m_2 \equiv K_5 + K_7\,,\\
\tilde{\lambda}_{1,j} &= \frac{u_{2,j}}{g}\,, \qquad j = 1,\ldots,K_2 \equiv n_1\,,\\
\tilde{\lambda}_{2,j} &= \frac{u_{6,j}}{g}\,, \qquad j = 1,\ldots,K_6 \equiv n_2\,.
\end{aligned} \tag{4.8}$$

Assuming [8]

$$S_0(\lambda_k\,, \lambda_j)^2 = \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)\left(\frac{1 - \frac{1}{x_j^+ x_k^-}}{1 - \frac{1}{x_j^- x_k^+}}\right)\sigma(\lambda_j, \lambda_k)^2 \tag{4.9}$$

and recalling [15]

$$x^{\pm}(-\lambda) = -x^{\mp}(\lambda)\,, \tag{4.10}$$

[7] We identify the variables $x_j^{\pm}$, $z_{\alpha,j}^{-}$ and $\tilde{\lambda}_{\alpha,j}$ in [26] with $x^{\pm}(p_j)$, $x^{+}(\lambda_j^{(\alpha)})$ and $\tilde{u}_j^{(\alpha)}$ in [7], respectively.

[8] This expression differs from the one given by Eq. (36) in [7] by the interchange $j \leftrightarrow k$; however, it seems to be consistent with the conventions in [26].

the first equation (4.1) becomes

$$
\begin{aligned}
e^{-2i\lambda_k(L+K_4+\frac{K_1-K_3+K_7-K_5}{2})}\Phi(\lambda_k) = \prod_{\substack{j=1\\ j\neq k}}^{K_4} \Bigg\{ & \left(\frac{x^-_{4,k}-x^+_{4,j}}{x^+_{4,k}-x^-_{4,j}}\right)\left(\frac{1-\frac{g^2}{x^+_{4,j}x^-_{4,k}}}{1-\frac{g^2}{x^-_{4,j}x^+_{4,k}}}\right)\sigma(\lambda_j,\lambda_k)^2 \\
& \times \left(\frac{x^-_{4,k}+x^-_{4,j}}{x^+_{4,k}+x^+_{4,j}}\right)\left(\frac{1+\frac{g^2}{x^-_{4,j}x^-_{4,k}}}{1+\frac{g^2}{x^+_{4,j}x^+_{4,k}}}\right)\sigma(-\lambda_k,\lambda_j)^2 \Bigg\} \\
& \times \prod_{j=1}^{K_3}\left(\frac{x^+_{4,k}-x_{3,j}}{x^-_{4,k}-x_{3,j}}\right)\left(\frac{x^+_{4,k}+x_{3,j}}{x^-_{4,k}+x_{3,j}}\right)\prod_{j=1}^{K_5}\left(\frac{x^+_{4,k}-x_{5,j}}{x^-_{4,k}-x_{5,j}}\right)\left(\frac{x^+_{4,k}+x_{5,j}}{x^-_{4,k}+x_{5,j}}\right) \\
& \times \prod_{j=1}^{K_1}\left(\frac{1-\frac{g^2}{x_{1,j}x^+_{4,k}}}{1-\frac{g^2}{x_{1,j}x^-_{4,k}}}\right)\left(\frac{1+\frac{g^2}{x_{1,j}x^+_{4,k}}}{1+\frac{g^2}{x_{1,j}x^-_{4,k}}}\right)\prod_{j=1}^{K_7}\left(\frac{1-\frac{g^2}{x_{7,j}x^+_{4,k}}}{1-\frac{g^2}{x_{7,j}x^-_{4,k}}}\right)\left(\frac{1+\frac{g^2}{x_{7,j}x^+_{4,k}}}{1+\frac{g^2}{x_{7,j}x^-_{4,k}}}\right), \\
& k=1,\ldots,K_4\,.
\end{aligned}
\tag{4.11}
$$

With the help of the definitions [7]

$$
u_{i,j} = x_{i,j} + \frac{g^2}{x_{i,j}}\,, \qquad i = 1,3,5,7\,, \tag{4.12}
$$

the second set of equations (4.2) becomes

$$
\begin{aligned}
&\prod_{j=1}^{K_4}\left(\frac{1-\frac{g^2}{x_{1,k}x^+_{4,j}}}{1-\frac{g^2}{x_{1,k}x^-_{4,j}}}\right)\left(\frac{1+\frac{g^2}{x_{1,k}x^-_{4,j}}}{1+\frac{g^2}{x_{1,k}x^+_{4,j}}}\right) = \prod_{j=1}^{K_2}\left(\frac{u_{1,k}-u_{2,j}-\frac{i}{2}}{u_{1,k}-u_{2,j}+\frac{i}{2}}\right)\left(\frac{u_{1,k}+u_{2,j}-\frac{i}{2}}{u_{1,k}+u_{2,j}+\frac{i}{2}}\right),\ k=1,\ldots,K_1\,, \\
&\prod_{j=1}^{K_4}\left(\frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}\right)\left(\frac{x_{3,k}+x^-_{4,j}}{x_{3,k}+x^+_{4,j}}\right) = \prod_{j=1}^{K_2}\left(\frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}}\right)\left(\frac{u_{3,k}+u_{2,j}-\frac{i}{2}}{u_{3,k}+u_{2,j}+\frac{i}{2}}\right),\ k=1,\ldots,K_3\,, \\
&\prod_{j=1}^{K_4}\left(\frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}\right)\left(\frac{x_{5,k}+x^-_{4,j}}{x_{5,k}+x^+_{4,j}}\right) = \prod_{j=1}^{K_6}\left(\frac{u_{5,k}-u_{6,j}-\frac{i}{2}}{u_{5,k}-u_{6,j}+\frac{i}{2}}\right)\left(\frac{u_{3,k}+u_{6,j}-\frac{i}{2}}{u_{5,k}+u_{6,j}+\frac{i}{2}}\right),\ k=1,\ldots,K_5\,, \\
&\prod_{j=1}^{K_4}\left(\frac{1-\frac{g^2}{x_{7,k}x^+_{4,j}}}{1-\frac{g^2}{x_{7,k}x^-_{4,j}}}\right)\left(\frac{1+\frac{g^2}{x_{7,k}x^-_{4,j}}}{1+\frac{g^2}{x_{7,k}x^+_{4,j}}}\right) = \prod_{j=1}^{K_6}\left(\frac{u_{7,k}-u_{6,j}-\frac{i}{2}}{u_{7,k}-u_{6,j}+\frac{i}{2}}\right)\left(\frac{u_{7,k}+u_{6,j}-\frac{i}{2}}{u_{7,k}+u_{6,j}+\frac{i}{2}}\right),\ k=1,\ldots,K_7\,.
\end{aligned}
\tag{4.13}
$$

Finally, the third set of equations (4.3) becomes

$$\prod_{j=1}^{K_1}\left(\frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}}\right)\left(\frac{u_{2,k}+u_{1,j}+\frac{i}{2}}{u_{2,k}+u_{1,j}-\frac{i}{2}}\right)\prod_{j=1}^{K_3}\left(\frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\right)\left(\frac{u_{2,k}+u_{3,j}+\frac{i}{2}}{u_{2,k}+u_{3,j}-\frac{i}{2}}\right)$$
$$=\prod_{\substack{j=1\\ j\neq k}}^{K_2}\left(\frac{u_{2,k}-u_{2,j}+i}{u_{2,k}-u_{2,j}-i}\right)\left(\frac{u_{2,k}+u_{2,j}+i}{u_{2,k}+u_{2,j}-i}\right)\,,\quad k=1,\ldots,K_2\,,$$

$$\prod_{j=1}^{K_5}\left(\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\right)\left(\frac{u_{6,k}+u_{5,j}+\frac{i}{2}}{u_{6,k}+u_{5,j}-\frac{i}{2}}\right)\prod_{j=1}^{K_7}\left(\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}}\right)\left(\frac{u_{6,k}+u_{7,j}+\frac{i}{2}}{u_{6,k}+u_{7,j}-\frac{i}{2}}\right)$$
$$=\prod_{\substack{j=1\\ j\neq k}}^{K_6}\left(\frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i}\right)\left(\frac{u_{6,k}+u_{6,j}+i}{u_{6,k}+u_{6,j}-i}\right)\,,\quad k=1,\ldots,K_6\,. \tag{4.14}$$

As expected, the open-chain BAEs (4.13), (4.14) are precisely "doubled" with respect to the corresponding closed-chain results given by Eqs. (52), (53) in [7], respectively. As we shall see below, Eqs. (4.14) are not relevant for the scalar sector.

4.2 Weak-coupling limit

We perform the weak-coupling ($g \to 0$) limit by setting

$$x_{i,j}\to\frac{u_{i,j}}{g}\,,\quad i=1,3,5,7\,,\qquad x^{\pm}_{4,j}\to\frac{1}{g}\left(u_{4,j}\pm\frac{i}{2}\right)\,, \tag{4.15}$$

and keeping the u's finite. Since

$$e^{i\lambda_k}=\frac{x^+_{4,k}}{x^-_{4,k}}\to\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\,, \tag{4.16}$$

the first equation (4.11) becomes

$$\begin{aligned}\left(\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\right)^{2L+2K_4+K_1-K_3+K_7-K_5}\Phi(\lambda_k)^{-1} &= \prod_{\substack{j=1\\ j\neq k}}^{K_4}\left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\right)\left(\frac{u_{4,k}+u_{4,j}+i}{u_{4,k}+u_{4,j}-i}\right)\\ &\times\prod_{j=1}^{K_3}\left(\frac{u_{4,k}-u_{3,j}-\frac{i}{2}}{u_{4,k}-u_{3,j}+\frac{i}{2}}\right)\left(\frac{u_{4,k}+u_{3,j}-\frac{i}{2}}{u_{4,k}+u_{3,j}+\frac{i}{2}}\right)\\ &\times\prod_{j=1}^{K_5}\left(\frac{u_{4,k}-u_{5,j}-\frac{i}{2}}{u_{4,k}-u_{5,j}+\frac{i}{2}}\right)\left(\frac{u_{4,k}+u_{5,j}-\frac{i}{2}}{u_{4,k}+u_{5,j}+\frac{i}{2}}\right)\,.\end{aligned} \tag{4.17}$$

Moreover, according to (4.5), the factor $\Phi(\lambda_k)$ becomes

$$\Phi(\lambda_k) \to \left(\frac{u_{4,k}+\frac{i}{2}}{u_{4,k}-\frac{i}{2}}\right)^4 \,, \tag{4.18}$$

assuming $k_0^{\pm}(\lambda) \to 1$, as suggested by [12, 15].

In the weak-coupling limit, the second set of equations (4.13) becomes

$$\begin{aligned}
\prod_{j=1}^{K_4}\left(\frac{u_{3,k}-u_{4,j}-\frac{i}{2}}{u_{3,k}-u_{4,j}+\frac{i}{2}}\right)\left(\frac{u_{3,k}+u_{4,j}-\frac{i}{2}}{u_{3,k}+u_{4,j}+\frac{i}{2}}\right) &= \prod_{j=1}^{K_2}\left(\frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}}\right)\left(\frac{u_{3,k}+u_{2,j}-\frac{i}{2}}{u_{3,k}+u_{2,j}+\frac{i}{2}}\right) \,, \\
\prod_{j=1}^{K_2}\left(\frac{u_{1,k}-u_{2,j}-\frac{i}{2}}{u_{1,k}-u_{2,j}+\frac{i}{2}}\right)\left(\frac{u_{1,k}+u_{2,j}-\frac{i}{2}}{u_{1,k}+u_{2,j}+\frac{i}{2}}\right) &= 1 \,, \\
\prod_{j=1}^{K_4}\left(\frac{u_{5,k}-u_{4,j}-\frac{i}{2}}{u_{5,k}-u_{4,j}+\frac{i}{2}}\right)\left(\frac{u_{5,k}+u_{4,j}-\frac{i}{2}}{u_{5,k}+u_{4,j}+\frac{i}{2}}\right) &= \prod_{j=1}^{K_6}\left(\frac{u_{5,k}-u_{6,j}-\frac{i}{2}}{u_{5,k}-u_{6,j}+\frac{i}{2}}\right)\left(\frac{u_{5,k}+u_{6,j}-\frac{i}{2}}{u_{5,k}+u_{6,j}+\frac{i}{2}}\right) \,, \\
\prod_{j=1}^{K_6}\left(\frac{u_{7,k}-u_{6,j}-\frac{i}{2}}{u_{7,k}-u_{6,j}+\frac{i}{2}}\right)\left(\frac{u_{7,k}+u_{6,j}-\frac{i}{2}}{u_{7,k}+u_{6,j}+\frac{i}{2}}\right) &= 1 \,.
\end{aligned} \tag{4.19}$$

The third set of equations (4.14) remain the same in the weak-coupling limit.

4.3 Reduction to the $SO(6)$ sector

Following [2], we reduce the above weak-coupling BAEs to the scalar sector by first transforming to the "beauty" form, and then removing the roots $u_{1,k}\,, u_{2,k}\,, u_{6,k}\,, u_{7,k}$. The latter procedure corresponds to removing the outer two nodes on each side of the $su(2,2|4)$ Dynkin diagram, leaving just the three nodes of $so(6) = su(4)$. In terms of the notation (2.37), it follows that

$$\begin{aligned}
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_3} e_2(u_{3,k}-u_{3,j})\, e_2(u_{3,k}+u_{3,j}) \\
&\quad \times \prod_{j=1}^{K_4} e_{-1}(u_{3,k}-u_{4,j})\, e_{-1}(u_{3,k}+u_{4,j}) \,, \quad k = 1\,, \ldots\,, K_3\,,
\end{aligned}$$

$$
\begin{aligned}
e_1(u_{4,k})^{2L+2K_4-4-K_3-K_5} &= \prod_{\substack{j=1\\ j\neq k}}^{K_4} e_2(u_{4,k}-u_{4,j})\, e_2(u_{4,k}+u_{4,j}) \\
&\quad \times \prod_{j=1}^{K_3} e_{-1}(u_{4,k}-u_{3,j})\, e_{-1}(u_{4,k}+u_{3,j}) \\
&\quad \times \prod_{j=1}^{K_5} e_{-1}(u_{4,k}-u_{5,j})\, e_{-1}(u_{4,k}+u_{5,j})\,, \quad k=1\,,\ldots\,,K_4\,,
\end{aligned}
$$

$$
\begin{aligned}
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_5} e_2(u_{5,k}-u_{5,j})\, e_2(u_{5,k}+u_{5,j}) \\
&\quad \times \prod_{j=1}^{K_4} e_{-1}(u_{5,k}-u_{4,j})\, e_{-1}(u_{5,k}+u_{4,j})\,, \quad k=1\,,\ldots\,,K_5\,.
\end{aligned} \tag{4.20}
$$

Comparing these BAEs with those which we have proposed, we see that, upon identifying $u_{3,k}\,, u_{4,k}\,, u_{5,k}$ in (4.20) with $u_{2,k}\,, u_{1,k}\,, u_{3,k}$ in (3.1), respectively, the equations almost match. The only difference is in the exponent of the term corresponding to the "massive" node: in (3.1) it is $2L+2$, while in (4.20) it is $2L+2K_4-4-K_3-K_5$. This sort of mismatch also occurs in the periodic case [7]. Evidently, the exact exponent cannot be deduced from an analysis (such as [26]) which is based solely on S-matrices, and requires additional input, such as the weak-coupling result proposed here.

5 Discussion

We have investigated the open spin chain describing the scalar sector of the $Y=0$ giant graviton brane at weak coupling. We have provided a direct proof of integrability in the $SU(3)$ and $SU(2)$ sectors by constructing the transfer matrices, namely, (2.23) and (A.7), respectively. Expanding the transfer matrix $t(u)$ in powers of u generates the Hamiltonian and the higher local conserved quantities. We have determined the eigenvalues of these transfer matrices in terms of roots of the corresponding BAEs, namely, (2.38) and (A.11), respectively. Based on these results, we have proposed BAEs for the full $SO(6)$ sector (3.1). Finally, we have found that, in the weak-coupling limit, the recently-proposed all-loop BAEs [26] essentially agree with those which we have proposed.

There are evidently several outstanding questions which remain to be addressed. It would be interesting to construct the transfer matrix for the full $SO(6)$ sector and check directly the proposed BAEs (3.1). In that case, the BYBEs for the K-matrices are significantly more

complicated than in the $SU(3)$ case, and we have not yet succeeded to find appropriate solutions. Perhaps it may be feasible to investigate other sectors, as well as finite-size corrections. Finally, it would also be interesting to investigate the case of the $Z = 0$ brane, for which there are boundary degrees of freedom.

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A The $SU(2)$ sector

We briefly consider here the $SU(2)$ subsector of the $Y = 0$ brane. This sector consists of only fields Z and Y. The Hamiltonian is again given by (2.2) - (2.4), where the permutation matrix is given by (2.5) with $n = 2$. Choosing the basis $|Z\rangle = \binom{1}{0}$, $|Y\rangle = \binom{0}{1}$, the projector Q^Y is now given by (cf. (2.6))

$$Q^Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{A.1}$$

The left boundary term can be factorized as follows

$$H^L_{bt} = Q^Y_0 \, (\mathbb{I} - \mathcal{P}_{0,1}) \, Q^Y_0 = Q^Y_0 (\mathbb{I} - Q^Y_1) = Q^Y_0 \, q^Y_1 \, , \tag{A.2}$$

where q^Y is defined as [15]

$$q^Y = \mathbb{I} - Q^Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{A.3}$$

The right boundary term can be expressed in a similar fashion,

$$H^R_{bt} = q^Y_L \, Q^Y_{L+1} \, . \tag{A.4}$$

As noted in [15], since the fields at sites 0 and $L+1$ cannot be Y's, they must be Z's. That is, the spins at sites 0 and $L+1$ are fixed, and can henceforth be ignored. The Hilbert space is therefore simply (cf. (2.1))

$$\overset{\overset{1}{\downarrow}}{C^2} \otimes \cdots \otimes \overset{\overset{L}{\downarrow}}{C^2} \, , \tag{A.5}$$

and the Hamiltonian becomes

$$H = 2g^2 \left(q_1^Y + \sum_{l=1}^{L-1} h_{l,l+1} + q_L^Y \right) . \tag{A.6}$$

The Hamiltonian (A.6) is that of an open XXX (isotropic) spin-1/2 chain with diagonal boundary terms, which has been well studied. The commuting transfer matrix is given by [30]

$$t(u) = \mathrm{tr}_a \, K_a^L(u) \, T_{a1\cdots L}(u) \, K_a^R(u) \, \hat{T}_{a1\cdots L}(u) \, , \tag{A.7}$$

where the monodromy matrices are as before (2.11), (2.25), and the K-matrices are given by

$$K^R(u) = \begin{pmatrix} i+u & 0 \\ 0 & i-u \end{pmatrix} , \qquad K^L(u) = \begin{pmatrix} -2i-u & 0 \\ 0 & u \end{pmatrix} . \tag{A.8}$$

The relation of this transfer matrix to the Hamiltonian (A.6) is again given by (2.27) where now

$$c_1 = 2g^2 \left(\frac{i}{4} (-1)^{L+1} \right) , \qquad c_2 = 2g^2 \left(L + \frac{1}{2} \right) . \tag{A.9}$$

The eigenvalues of the transfer matrix (A.7) are given by

$$\Lambda(u) = -\frac{2}{(2u+i)} \left\{ (u+i)^{2L+3} \frac{Q_1(u-i/2)}{Q_1(u+i/2)} + u^{2L+3} \frac{Q_1(u+3i/2)}{Q_1(u+i/2)} \right\} , \tag{A.10}$$

where $Q_1(u)$ is given by (2.34) with $j = 1$. Analyticity of $\Lambda(u)$ at $u = u_{1,k} - i/2$ leads to the BAEs

$$e_1(u_{1,k})^{2L+2} = \prod_{\substack{j=1 \\ j \neq k}}^{m_1} e_2(u_{1,k} - u_{1,j}) \, e_2(u_{1,k} + u_{1,j}) \, , \quad k = 1 \, , \ldots \, , m_1 \, . \tag{A.11}$$

Finally, we note that the eigenvalues of the Hamiltonian (A.6) are given in terms of the Bethe roots by the same relation (2.39).

B Numerical results for the $SU(3)$ sector

We have verified the completeness of our Bethe ansatz solution in the $SU(3)$ sector (2.33) - (2.39) for $L = 1, 2, 3$. Our results for the energies (E), spins (s), and Bethe roots ($\{u_{1,k}\}$, $\{u_{2,k}\}$) are presented in Tables 1 and 2.

Solving BAEs directly is notoriously difficult. We have obtained the Bethe roots by instead using "McCoy's method" (see, e.g., [39, 40, 41]). The basic idea is to work backwards: one first explicitly computes the eigenvalues $\Lambda(u)$ as polynomials in u by numerically diagonalizing the transfer matrix; and then one solves the $T-Q$ equations (2.33) for $Q_j(u)$, which are also polynomials in u. Finally, one finds the zeros of $Q_j(u)$, which are the sought-after Bethe roots $u_{j,k}$. This method therefore produces solutions of the BAEs without actually solving the equations! Although this method is inconvenient for determining the eigenvalues of an integrable Hamiltonian (direct diagonalization is much faster), it is ideal for determining all the Bethe roots – and therefore checking the completeness of a Bethe ansatz solution – for small values of L.

We have verified that the energies, computed from the Bethe roots using (2.39), coincide with the result obtained by direct diagonalization of the Hamiltonian (2.2).

For each value of $SU(2)$ spin s, there is a $(2s+1)$-fold degeneracy. Taking into account this degeneracy, we find in total $2^2 3^L$ levels for each value of L, which coincides with the dimension of the Hilbert space (2.1). In this way, we verify the completeness of the Bethe ansatz solution.

Finally, it may be worth noting that the tables reveal many "accidental" degeneracies: levels described by different sets of Bethe roots having the same energy. This suggests that the Hamiltonian may have some interesting higher symmetry.

L	$E/(2g^2)$	s	$\{u_{1,k}\}$	$\{u_{2,k}\}$
1	0	3/2	–	–
	1	1/2	$\sqrt{3}/2$	0
	2	1	1/2	–
	2	0	$\pm i/2$	0
	3	1/2	$1/(2\sqrt{3})$	0
2	0	2	–	–
	0.585786	1	1.20711	0
	1	3/2	$\sqrt{3}/2$	–
	1.26795	0, 1/2	$0.716015 \pm 0.512521i$	0
	2	1	1/2	0
	2	1	$\pm i/2$	–
	2	0, 1/2	$\pm i/2$	0
	3	3/2	$1/(2\sqrt{3})$	–
	3.41421	1	0.207107	0
	4	1/2	$1/(2\sqrt{3})\,,\sqrt{3}/2$	$\sqrt{2/3}$
	4.73205	0, 1/2	0.230955, 0.668326	0

Table 1: Energy, spin, and Bethe roots for $L = 1\,,2$.

$E/(2g^2)$	s	$\{u_{1,k}\}$	$\{u_{2,k}\}$
0	5/2	–	–
0.381966	3/2	1.53884	0
0.585786	2	1.20711	–
0.82259	1/2, 1	$1.11504 \pm 0.545054i$	0
1.07919	0	$0.709462\,, 0.695742 \pm 0.987839i$	$0\,, 1.56857i$
1.26795	3/2	$0.716015 \pm 0.512521i$	–
1.38197	3/2	0.688191	0
1.38197	1/2	$\pm i/2\,, 1.36676i$	0
1.58579	1/2, 1	$0.513712 \pm 0.499602i$	0
1.69722	1/2	$\pm i/2\,, 1.88488i$	0
2	2	1/2	–
2	1, 3/2	$\pm i/2$	–
2	0, 1	$\pm i/2$	0
2.58579	1	$0.5\,, 1.20711$	1.0505
2.61803	3/2	0.363271	0
3	0	$0.661848\,, 0.531587 \pm 0.501604i$	$0\,, i/\sqrt{2}$
3	1/2	$\pm i/2\,, \sqrt{3}/2$	1
3.31526	0	$0.665753\,, 0.26039 \pm 0.500009i$	$0\,, 1.15861i$
3.32164	1/2, 1	$0.396968\,, 0.949669$	0
3.41421	2	0.207107	–
3.61803	1/2	$\pm i/2\,, 0.606658$	0
3.61803	3/2	0.16246	0
4	1	$(\pm 1 + \sqrt{2})/2$	1
4.41421	1/2, 1	$0.17238\,, 0.970407$	0
4.68474	0	$0.221275\,, 0.667076 \pm 0.508802i$	$0\,, 0.810943i$
4.73205	3/2	$0.230955\,, 0.668326$	–
5	1/2	$\pm i/2\,, 1/(2\sqrt{3})$	$\sqrt{5}/3$
5	0	$0.212263\,, 0.478697 \pm 0.501676i$	$0\,, i\sqrt{3/2}$
5.30278	1/2	$\pm i/2\,, 0.229729$	0
5.41421	1	$0.207107\,, 0.5$	0.62964
5.85577	1/2, 1	$0.179337\,, 0.427295$	0
6.92081	0	$0.176138\,, 0.409803\,, 0.883877$	$0\,, 0.678531$

Table 2: Energy, spin, and Bethe roots for $L = 3$.

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